International *Mathematica* User Conference 2008

Casimir force computations in non-trivial geometries using *Mathematica*

Fabrizio Pinto

InterStellar Technologies Corporation

Properties of the Vacuum. 1. Mechanical and Thermodynamic (1983)

Reference: J. Ambjørn and S. Wolfram, *Annals of Physics* 147 (1983) 1-32.

Abstract

Casimir energies are calculated for quantized fields in cavities with a variety of forms. Consequences for models of the vacuum state are considered. The possibility of negative mass systems is discussed. Results on energy and entropy of finite quantum systems at non-zero temperature are given.

next

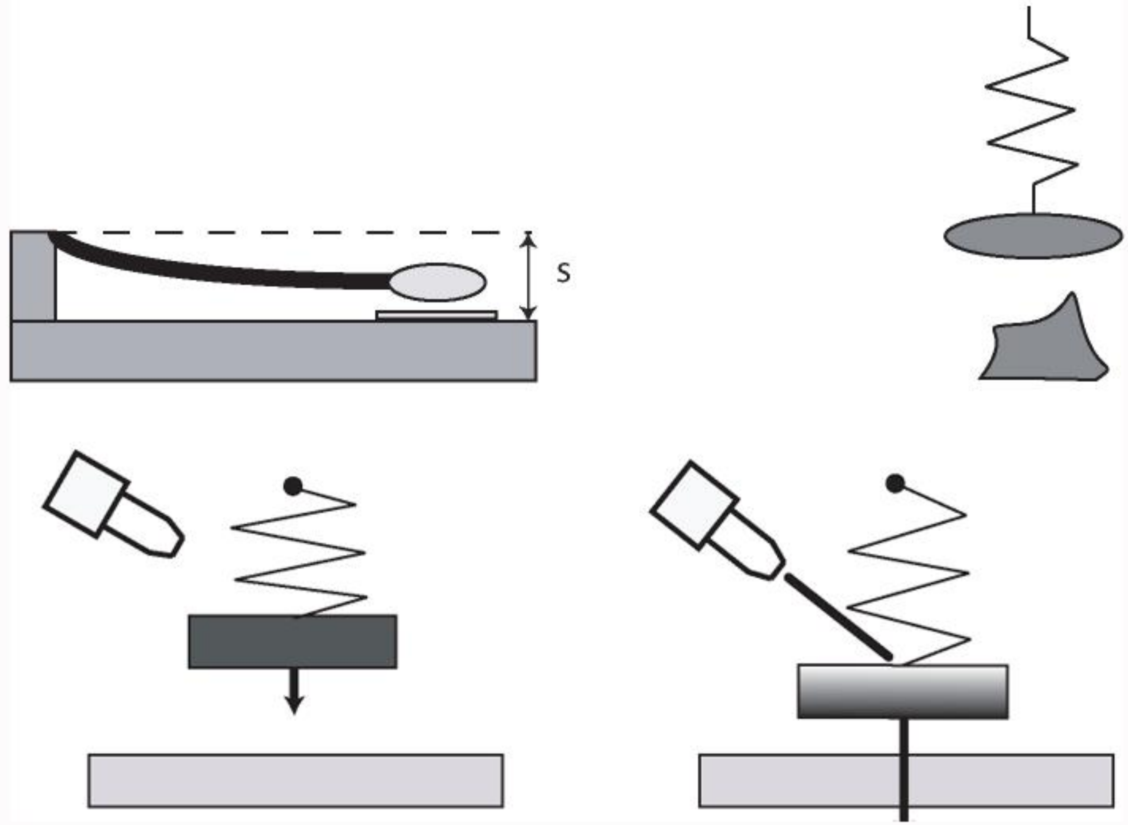
Given the van der Waals forces between a pair of particles, a simple scheme would take the Casimir forces between surfaces as a superposition of forces between their constituent particles. A conducting surface may be taken to consist of a collection of polarizable particles. According to the simple scheme, the Casimir forces between conducting surfaces would then always be attractive. The results of Section 4 show that in practice they are often repulsive (as in the case of a cubical cavity). The simple scheme fails for essentially two reasons. First, the presence of the boundaries modifies the modes of the electromagnetic field, and thus changes the virtual photon propagator and the spectrum of the zero-point fluctuations in the electromagnetic field. Second, whenever the boundary is connected (as when $p > 1$), it is not possible to separate the ``self-energy'' of the boundary from the true Casimir energy of the confined field. In the case of two parallel planes, the simple scheme nevertheless leads to a correct Casimir energy. This result is probably fortuitous: the scheme is known to fail for $p > 1$ and probably fails with non-planar boundaries even with $p = 1$.

The failure of superposition for van der Waals forces is in principle amenable to experimental investigation. The repulsive nature of Casimir forces in a spherical cavity could perhaps be seen in the behaviour of small bubbles (possibly in liquid ^4He).

"I have heard statements that the role of academic research in innovation is slight. This is about the most blatant piece of nonsense it has been my fortune to stumble upon."

Hendrik Brugt Gerhard Casimir (1909-2000)

Science - Technology spiral
(H. B. G. Casimir, *Haphazard Reality - Half a Century of Science*)

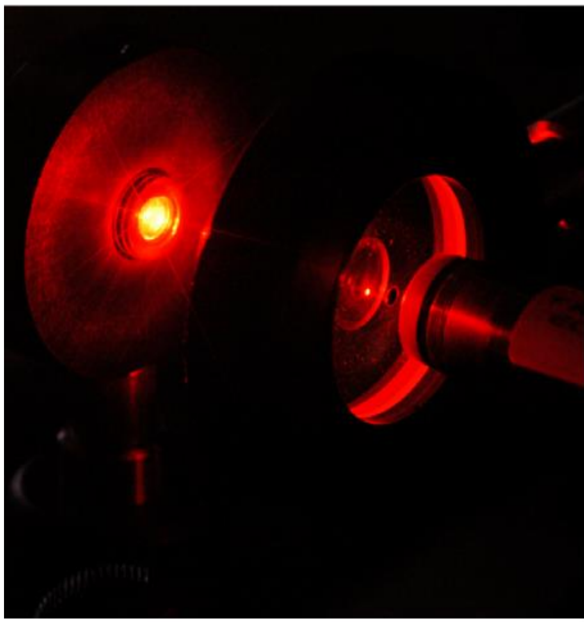


INTERSTELLAR TECHNOLOGIES CORPORATION, THE QUANTUM VACUUM ENGINEERING COMPANY

Company Objectives:

1. Develop a full suite of quantum vacuum engineering techniques to manipulate both the magnitude and the sign of van der Waals forces.

2. Apply van der Waals force manipulation to stiction remediation and to the design of marketable nano-devices capable to deliver disruptive performance advances and entirely novel capabilities.



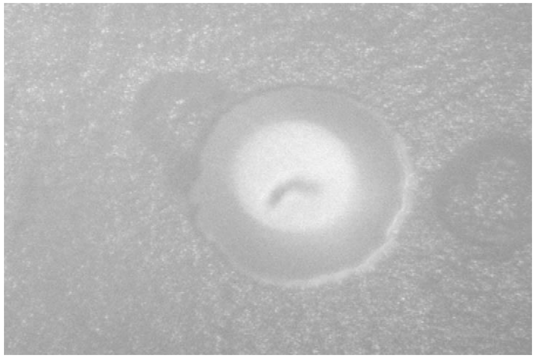
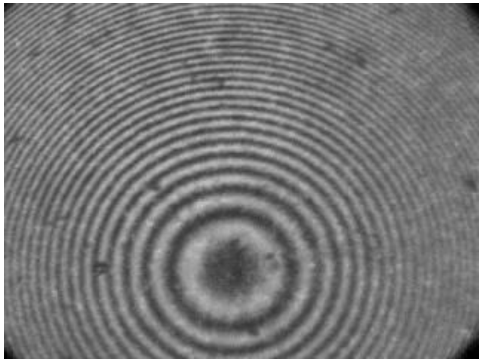
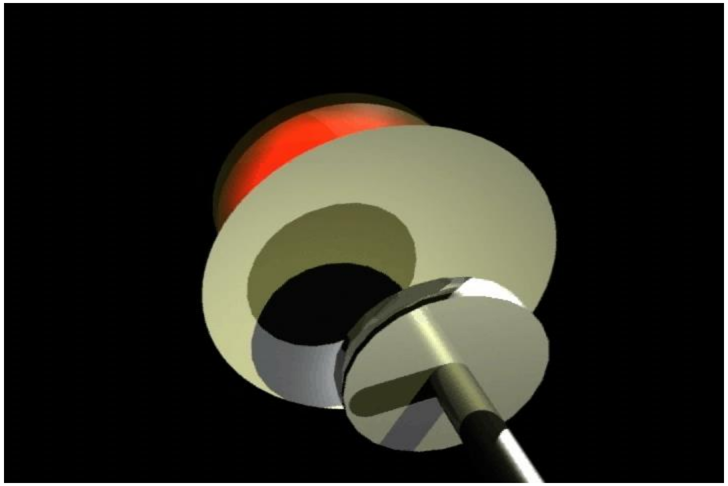
Selected Company Milestones and Activities:

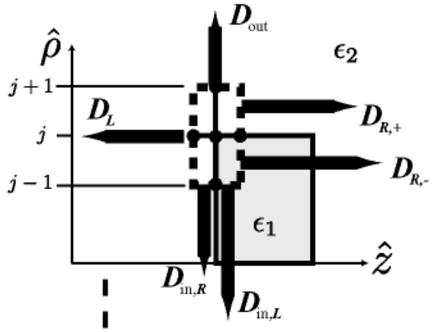
First privately owned company in the world uniquely devoted to industrial applications of van der Waals forces in nanotechnology (incorporated in 1999).

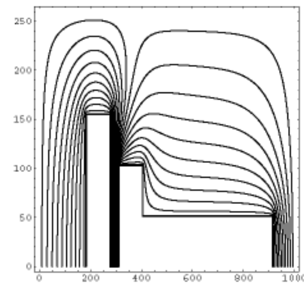
First accurate computation of van der Waals force manipulation in semiconductors. Invented the Casimir-force engine cycle for energy conversion and storage.

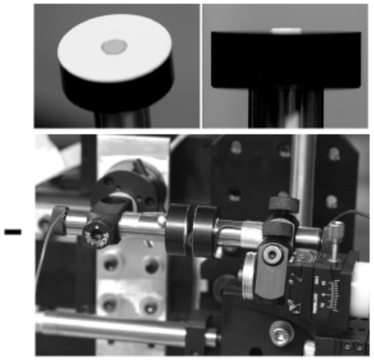
U.S. Patents: 6,477,028, 6,593,566, 6,650,527, 6,661,576, 6,665,167, 6,842,326, and 6,920,032; IP also in Israel, the EU, and Japan. TRANSVACER trademark (*TRANSducer of VACuum enERgy*).

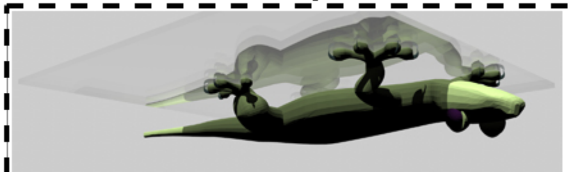
Publication record in the Physical Review, J. of Physics A, Int. J. Physics, American . J. of Physics, J. Sound & Vibration, etc.

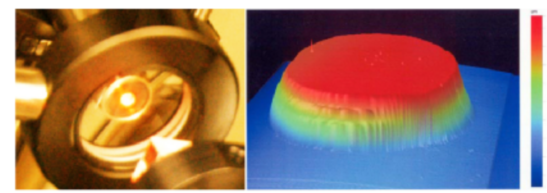




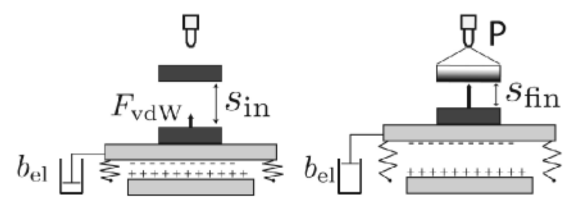












Technical Approach:

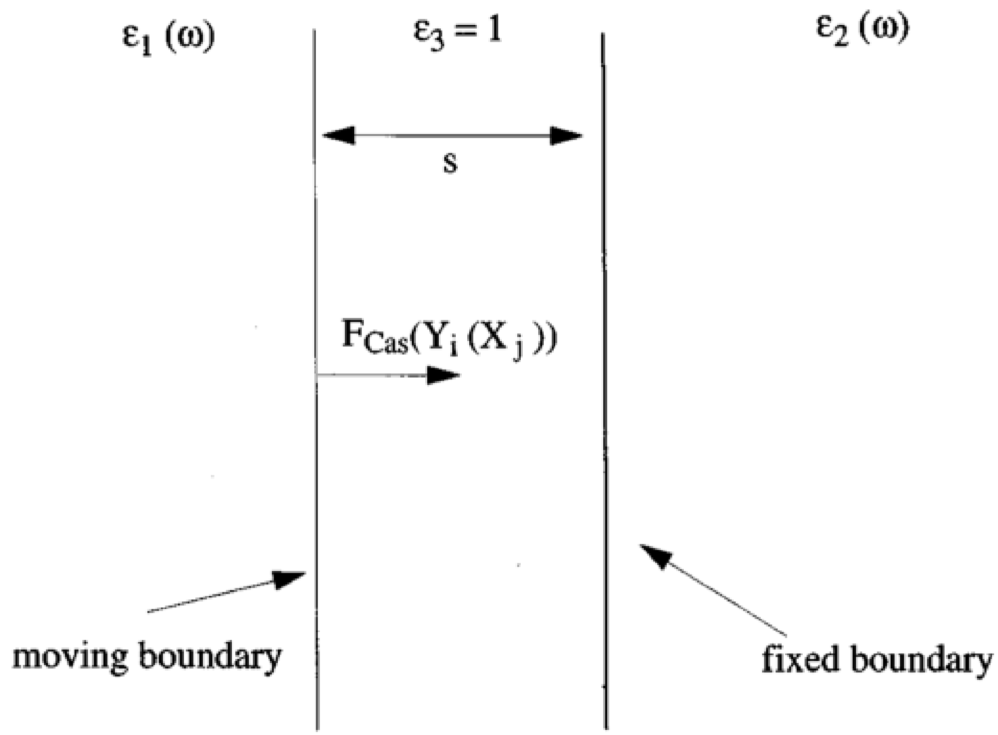
1. Develop advanced numerical techniques for van der Waals force computation in realistic geometries of industrial interest to replace the highly idealized treatments in the literature.

2. Conclusively document van der Waals force modulation by a substantial leap in accurate electrostatic calibration of the device including visco-elastic effects, and experimental validation.

3. Demonstrate the actuation of a micro-membrane by means of light-driven van der Waals force manipulation.

4. Integrate van der Waals force manipulation in the design of: (1) sensors, (2) adaptive optics, (3) NEMS oscillators, and (4) an energy harvester and storage system.

5. Demonstrate downward-scalability in actual MEMS fabrication.


ε1 (ω)
ε3 = 1
ε2 (ω)
s
FCas(Yi (Xj))
moving boundary
fixed boundary

$$\hat{\mathbf{E}}^{+}(\mathbf{r},\omega_R) = i\frac{\omega_R}{c}\hat{\mathbf{A}}^{+}(\mathbf{r},\omega_R)$$

$$\hat{\mathbf{B}}^{+}(\mathbf{r},\omega_R) = \nabla \times \hat{\mathbf{A}}^{+}(\mathbf{r},\omega_R)$$

$$\nabla \times \hat{\mathbf{E}}^{+}(\mathbf{r},\omega_R) = i\,\frac{\omega_R}{c}\,\hat{\mathbf{B}}^{+}(\mathbf{r},\omega_R)$$

$$\nabla \times \hat{\mathbf{B}}^{+}(\mathbf{r},\omega_R) = -i\,\frac{\omega_R}{c}\,\hat{\mathbf{D}}^{+}(\mathbf{r},\omega_R) + \frac{4\pi}{c}\,\hat{\mathbf{J}}^{+}(\mathbf{r},\omega_R)$$

$$\hat{\mathbf{D}}^{+}(\mathbf{r},\omega_R) = \tilde{\epsilon}(\mathbf{r},\omega_R)\hat{\mathbf{E}}^{+}(\mathbf{r},\omega_R)$$

$$\nabla \times \nabla \times \hat{\mathbf{A}}^{+}(\mathbf{r},\omega_R) - \left(\frac{\omega_R}{c}\right)^2 \tilde{\epsilon}(\mathbf{r},\omega_R)\;\hat{\mathbf{A}}^{+}(\mathbf{r},\omega_R) = \frac{4\pi}{c}\,\hat{\mathbf{J}}^{+}(\mathbf{r},\omega_R)\,.$$

$$-\frac{\partial^2 \hat{A}_x^{+}(z,i\omega_I)}{\partial z^2} + \left(\frac{\omega_I}{c}\right)^2 \quad \tilde{\epsilon}(z,i\omega_I)\hat{A}_x^{+}(z,i\omega_I) \quad \frac{4\pi}{c}\,\hat{J}_x^{+}(z,i\omega_I)\,.$$

$$\hat{A}_x^{+}(z,i\omega_I) = \frac{1}{c}\int dz'\,G_x(z,z',i\omega_I)\hat{J}_x^{+}(z',i\omega_I)$$

$$-\left[\frac{\partial^2}{\partial z^2} - \left(\frac{\omega_I}{c}\right)^2 \tilde{\epsilon}(z,i\omega_I)\right] G_x(z,z',i\omega_I) = 4\pi\,\delta(z-z')$$

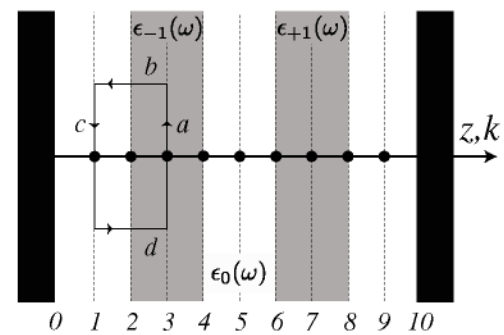
$$\hat{A}_x^{+}(k,i\omega_I) = \frac{1}{c}\sum_{k'} \Delta z\,G_x(k,k',i\omega_I)\hat{J}_x^{+}(k',i\omega_I)$$

$$\tilde{\mathbf{L}}_{\Delta z}\hat{A}_x^{+}(k,i\omega_I) = \frac{4\pi}{c}\,\hat{J}_x^{+}(k,i\omega_I)$$

$$\tilde{\mathbf{L}}_{\Delta z}G_x(k,k',i\omega_I) = \frac{1}{\Delta z}4\pi\,\mathbf{1}(k,k')$$

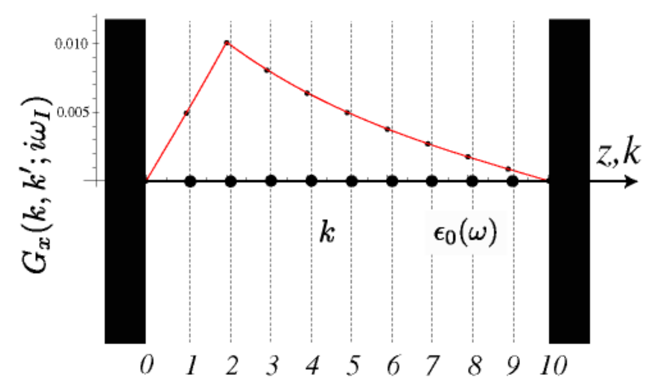
$$G_x(k,k',i\omega_I) = \frac{1}{\Delta z}4\pi\,\tilde{\mathbf{L}}_{\Delta z}^{-1}(k,k';i\omega_I)\cdot\mathbf{1}\,.$$

$$
\tilde{\mathbf{L}}(i\omega_I)\Big|_{\mathrm{empty}} = \begin{pmatrix}
2+\lambda_0^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2+\lambda_0^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2+\lambda_0^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2+\lambda_0^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2+\lambda_0^2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2+\lambda_0^2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2+\lambda_0^2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+\lambda_0^2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+\lambda_0^2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+\lambda_0^2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+\lambda_0^2
\end{pmatrix}
$$



$$
\tilde{\mathbf{L}}(i\omega_I)\Big|_{\mathrm{slabs}} = \begin{pmatrix}
2+\lambda_0^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2+\lambda_0^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2+\frac{1}{2}(\lambda_0^2+\lambda_{-1}^2) & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2+\lambda_{-1}^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2+\frac{1}{2}(\lambda_{-1}^2+\lambda_0^2) & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2+\lambda_0^2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2+\frac{1}{2}(\lambda_0^2+\lambda_{-1}^2) & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+\lambda_{-1}^2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+\frac{1}{2}(\lambda_{-1}^2+\lambda_0^2) & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+\lambda_0^2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2+\lambda_0^2
\end{pmatrix}
$$

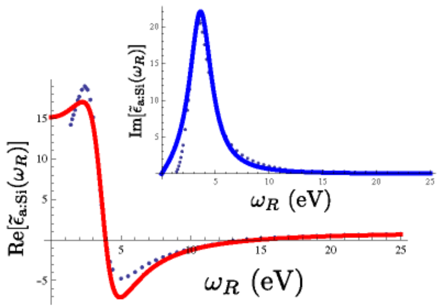
$$
-\frac{1}{(\Delta z)^2}\left[G_x(k+1,k') - 2G_x(k,k') + G_x(k-1,k')\right] + \left(\frac{\omega_I}{c}\right)^2 \tilde{\epsilon}(k,i\omega_I)G_x(k,k') = \frac{1}{\Delta z}4\pi\,\delta_{kk'}
$$



$$
\tilde{\mathbf{L}}_{\Delta z}(k,k';i\omega_I)G_x(k,k',i\omega_I) = 4\pi(\Delta z)\,\mathbf{1}
$$

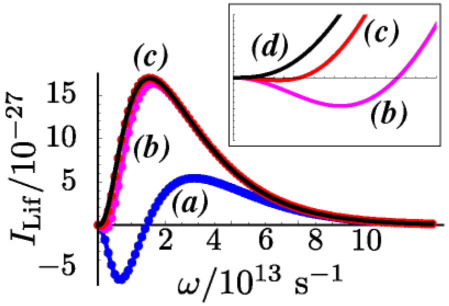
$$
G_x(z,z') = \frac{4\pi}{\Lambda^2 \sinh \Lambda^2 L} \times \begin{cases} \sinh \Lambda^2 z \sinh \Lambda^2 (z'-L); & z \le z' \\ \sinh \Lambda^2 z' \sinh \Lambda^2 (z-L); & z \ge z' \end{cases}
$$

$$\tilde{\epsilon}_{\mathrm{a:Si}}(i\omega_I) = 1 + \frac{\varepsilon_1}{1 + (\omega_I^2/\Omega_1)^2 + (\omega_I/\gamma)}$$

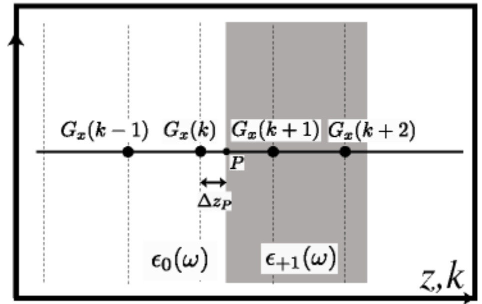


$$G_x(k_{LL}+1, k') - G_x(k_{LL}-1, k') = 0$$

$$G_x(k,k')\left[2 + (\Delta z)^2 \left(\frac{\omega_I}{c}\right)^2 \frac{\tilde{\epsilon}_0(i\omega_I) + \tilde{\epsilon}_{-1}(i\omega_I)}{2}\right]$$
$$-G_x(k-1,k') - G_x(k+1,k') = 4\pi(\Delta z)\delta_{kk'}\,.$$



$$\hat{A}_x^+(z)\Big|_{z_P} = \sum_{n=0}^{\infty} \frac{(\Delta z_P)^n}{n!} \frac{\partial^n \hat{A}_x^+(z)}{\partial z^n}\Big|_k$$



$$\frac{\partial \hat{A}_x^+(z)}{\partial z}\Big|_{z_P} = \sum_{n=0}^{\infty} \frac{(\Delta z_P)^n}{n!} \frac{\partial}{\partial z} \frac{\partial^n \hat{A}_x^+(z)}{\partial z^n}\Big|_{k\Delta z}$$

$$\overline{A}_x(z_P)\big|_L = \tilde{M}_{P,LL}(k)\,\overline{A}_x(k\Delta z)\,,$$

$$\tilde{M}_{P,LL} = \begin{pmatrix} 1 & -z_k + z_P & \frac{1}{2}(-z_k + z_P)^2 & \frac{1}{6}(-z_k + z_P)^3 & \frac{1}{24}(-z_k + z_P)^4 \\ 0 & 1 & -z_k + z_P & \frac{1}{2}(-z_k + z_P)^2 & \frac{1}{6}(-z_k + z_P)^3 \\ 0 & 0 & 1 & -z_k + z_P & \frac{1}{2}(-z_k + z_P)^2 \\ 0 & 0 & 0 & 1 & -z_k + z_P \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_x(z_P)|_L = A_x(z_P)|_R$$

$$\partial A_x(z_P)/\partial z|_L = \partial A_x(z_P)/\partial z|_R$$

$$\left.\frac{\partial^2 A_x(z_P)}{\partial z^2}\right|_R = \left.\frac{\partial^2 A_x(z_P)}{\partial z^2}\right|_L - \left(\frac{\omega_I}{c}\right)^2 (\tilde{\epsilon}_0 - \tilde{\epsilon}_{+1})\; A_x(z_P)|_L$$

$$\left.\frac{\partial^3 A_x(z_P)}{\partial z^3}\right|_R = \left.\frac{\partial^3 A_x(z_P)}{\partial z^3}\right|_L - \left(\frac{\omega_I}{c}\right)^2 (\tilde{\epsilon}_0 - \tilde{\epsilon}_{+1})\; \left.\frac{\partial A_x(z_P)}{\partial z}\right|_L$$

$$\left.\frac{\partial^4 A_x(z_P)}{\partial z^4}\right|_R = \left.\frac{\partial^4 A_x(z_P)}{\partial z^4}\right|_L - 2\left(\frac{\omega_I}{c}\right)^2 (\tilde{\epsilon}_0 - \tilde{\epsilon}_{+1})\; \left.\frac{\partial^2 A_x(z_P)}{\partial z^2}\right|_L + \left(\frac{\omega_I}{c}\right)^4 (\tilde{\epsilon}_0 - \tilde{\epsilon}_{+1})^2\; A_x(z_P)|_L\,.$$

$$\overline{A}_x(z_P)\big|_R = \tilde{M}_{LR}\, \overline{A}_x(z_P)\big|_L\ .$$

$$\tilde{M}_{LR} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{\omega_I^2}{c^2}(\tilde{\epsilon_0} - \tilde{\epsilon_1}) & 0 & 1 & 0 & 0 \\ 0 & -\frac{\omega_I^2}{c^2}(\tilde{\epsilon_0} - \tilde{\epsilon_1}) & 0 & 1 & 0 \\ \frac{\omega_I^4}{c^4}(\tilde{\epsilon_0} - \tilde{\epsilon_1})^2 & 0 & -\frac{2\omega_I^2}{c^2}(\tilde{\epsilon_0} - \tilde{\epsilon_1} & 0 & 1 \end{pmatrix}$$

$$\overline{A}_x((k+1)\Delta z) = \tilde{M}_P(k+1)\,\overline{A}_x(k\Delta z)$$

$$\tilde{M}_P(k+1) \equiv \tilde{M}_{P,LR} \cdot \tilde{M}_{LR} \cdot \tilde{M}_{P,LL}$$

$$\boxed{A_x(k+1) = \overline{V}_P^{\,+}\ \overline{A}_x(k)}$$

$$\overline{A}_x(k\Delta z) = \tilde{M}_P(k)\,\overline{A}_x((k+1)\Delta z)$$

$$\tilde{M}_P(k) \equiv \tilde{M}_{P,RR} \cdot \tilde{M}_{RL} \cdot \tilde{M}_{P,RL}$$

$$\boxed{A_x(k) = \overline{V}_P^{\,-}\ \overline{A}_x(k+1)}$$

$$C_L(k-1) = -1 + \frac{m_L[2z_P - \Delta z(2k-1)]}{6\Delta z}$$

$$C_L(k) = \tilde{\epsilon}_0 \frac{\omega_I^2}{c^2}(\Delta z)^2 + 2 - m_L + \tfrac{1}{6}m_L^2$$

$$C_L(k+1) = -1 - \frac{m_L[2z_P - \Delta z(2k-1)]}{6\Delta z}$$

$$m_L = \tfrac{1}{2}[z_P - (k+1)\Delta z]^2(\tilde{\epsilon}_0 - \tilde{\epsilon}_1)\frac{\omega_I^2}{c^2}$$

$$\tilde{\mathbf{L}}_{\Delta z}(k+1) = C_R(k)A_x(k) + C_R(k+1)A_x(k+1) + C_R(k+2)A_x(k+2)$$

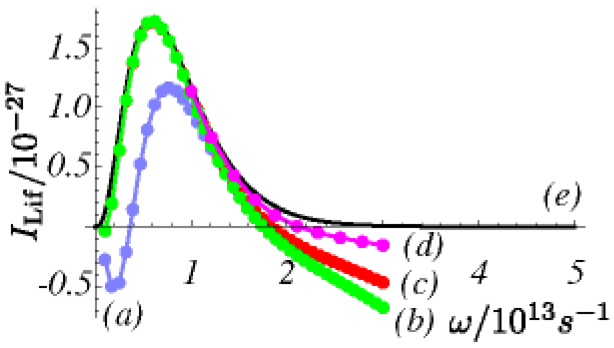
$$C_R(k) = -1 - \frac{m_R[2z_P - \Delta z(2k+3)]}{6\Delta z}$$

$$C_R(k+1) = \tilde{\epsilon}_1 \frac{\omega_I^2}{c^2}(\Delta z)^2 + 2 + m_R + \tfrac{1}{6}m_R^2$$

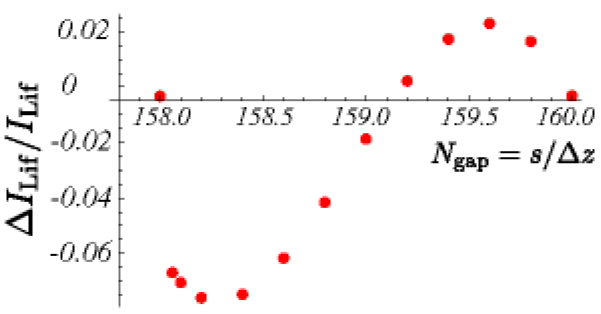
$$C_R(k+2) = -1 + \frac{m_R[2z_P - \Delta z(2k+3)]}{6\Delta z}$$

$$m_R = \tfrac{1}{2}(z_P - k\Delta z)^2(\tilde{\epsilon}_0 - \tilde{\epsilon}_1)\frac{\omega_I^2}{c^2}$$

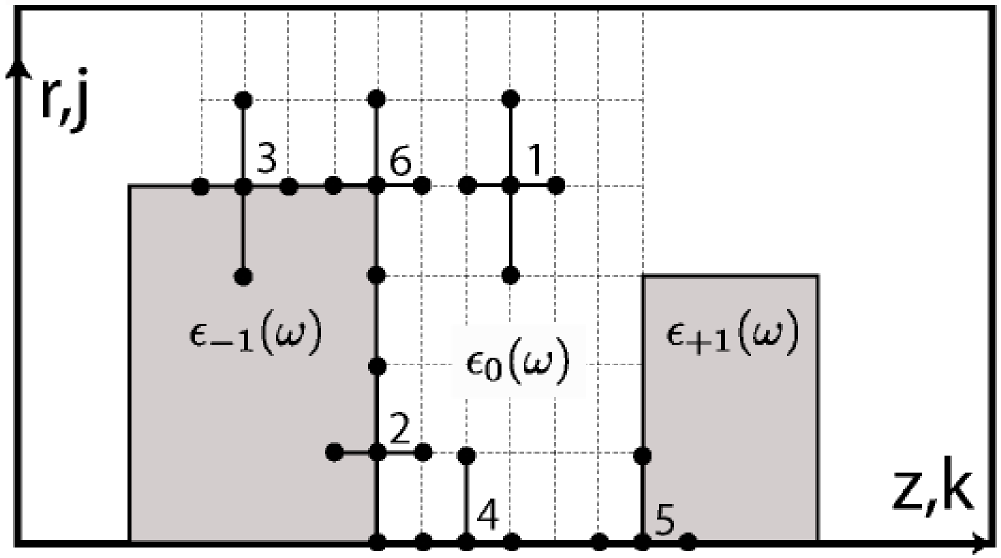
$$\tilde{\mathbf{L}}_{\Delta z}(k) = C_L(k-1)A_x(k-1) + C_L(k)A_x(k) + C_L(k+1)A_x(k+1)$$

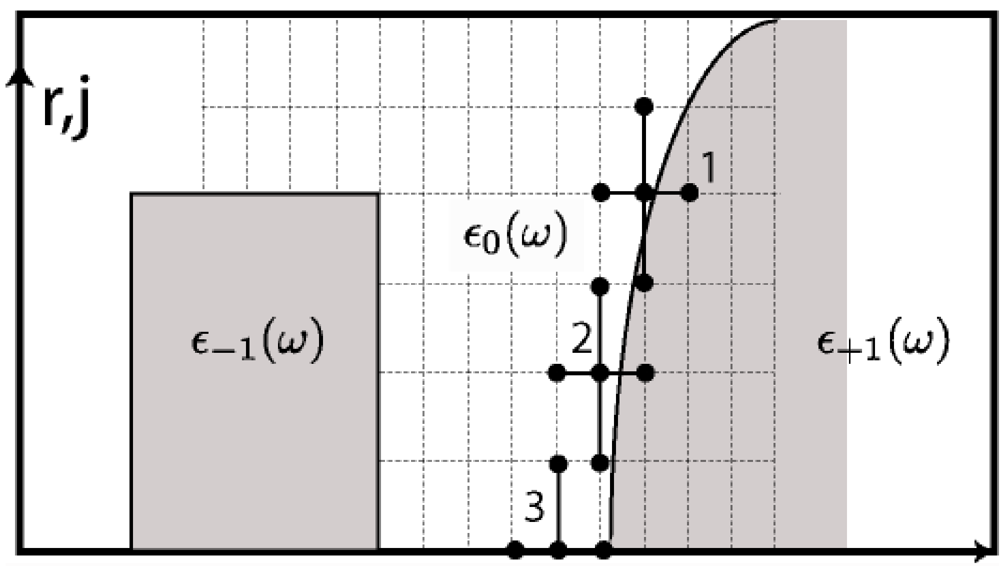


1.5
1.0
0.5
-0.5
$I_{\mathrm{Lif}}/10^{-27}$
1
2
4
5
(e)
(d)
(c)
(b)
(a)
$\omega/10^{13}s^{-1}$



0.02
0
-0.02
-0.04
-0.06
$\Delta I_{\mathrm{Lif}}/I_{\mathrm{Lif}}$
158.0
158.5
159.0
159.5
160.0
$N_{\mathrm{gap}} = s/\Delta z$


r,j
z,k
3
6
1
2
4
5
$\epsilon_{-1}(\omega)$
$\epsilon_0(\omega)$
$\epsilon_{+1}(\omega)$


r,j
1
2
3
$\epsilon_{-1}(\omega)$
$\epsilon_0(\omega)$
$\epsilon_{+1}(\omega)$

"The atoms move without interruption through all time. Some ... separate far from each other; the others maintain a vibrating motion either closely entangled with each other or confined by other atoms that have become entangled ... The degree of entanglement of the atoms determines the extent of the recoil from the collision."

letter to Herodotus by the Greek philosopher Epicurus (ca.341- 270 BC)

***"What seems to us the hardened and condensed
Must be of atoms among themselves more hooked ..."***

Lucretius (ca. 99 - ca.55 BC), writing in 50 BC

"I do not understand how these large globules of water stand out and hold themselves up, although I know for a certainty that it is not owing to any internal tenacity acting between the particles of water; whence it must follow that the cause of this effect is external."

Galileo Galilei, *Dialogue Concerning Two New Sciences* (1638)

"For I am induced by many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards each other and cohere in regular figures, or are repelled and recede from one another; which forces being unknown, philosophers have hitherto attempted the search of Nature in vain."

Sir Isaac Newton's own words in the Preface to his *Principia* (8 May 1686)