ATA

Advanced Tensor Analysis

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2013

Abstract

ATA is a tensor calculus package made for general purpose, easy to use with good looking formula display. This package was built for computing large tensor equations with the least amount of time. This package is unique in the sence that it allows you to input tensors in their tensor form and it has a simpe framework. The Output comes with tensor and derivatives of tensors only and any covariant derivatives are evaluated at the beginning. In addition, ATA uses the tensors symmetries to simplify an expression enormously. ATA can be of interest to the scientist working on any field where tensor calculus is required.

```
Clear["Global`*"];
Get["C:\\Library\\Mathematica\\Afak Publish\\ATA.m"];
```

First we must set the number of dimensions and name our coordinate.

```
SetSystem[3 , X ]
```

Tutorial

▪ Evaluating Tensor

Use the EvaExp Head to Evalaute your tensor and write in tensor form.

```
Aa // EvaExp
```

A_a

▪ ExpandIndex

Use the ExpandIndex Head to sum over repeated indices.

```
A_a A^a // EvaExp // ExpandIndex
```

$A^1 A_1 + A^2 A_2 + A^3 A_3$

ContractIndex

Use the ContractIndex Head to Contract repeated indices.

```
A_a A^a // EvaExp // ExpandIndex
A_a A^a // EvaExp // ExpandIndex // ContractIndex
```

$A^1 A_1 + A^2 A_2 + A^3 A_3$

$A^a A_a$

Define

Use the Define Head to Define your tensor .
Define[{Tensor Name,Tensor Indices,Tensor Definition,Symmetric Indices,Antisymmetric Indices,Tensor Elements}]
When using Define you must enter the tensor name and its indices, however its optional to enter the other elements such as "Symmetric Indices".
e.g. Define[{δ, {a,-b}}], defines δ^a_b
e.g. Define[{δ, {a,-b}, $g^{ac} g_{cb}$,{{a,-b}}, ,{{1,0,0},{0,1,0},{0,0,1}}}], defines $\delta^a_b = g^{ac} g_{cb}$ and δ^a_b is symmetric with respect to the permuation a and b.

```
Define[{W, {a, -b}, g^ac g_cb}]
W^a_b // EvaExp
```

$g^{ac} g_{cb}$

SimplifyTensor

SimplifyTensor Head uses elements four and five of Define[] to simplify equations
e.g Show that $T_{ab} + T_{ba}$=0 , where T_{ab} is antisymmetric

```
Define[{T, {-a, -b}, , , {{-a, -b}}}]
T_ab + T_ba // EvaExp
T_ab + T_ba // EvaExp // SimplifyTensor
```

$T_{ab} + T_{ba}$

0

$\mathcal{D}$ Operators

Use $\mathcal{D}$ Head when writing derivatives
e.g.
$V^a_{,b}$ is writing as $\mathcal{D}_1[V^a, \mathrm{b}]$
$V^a_{;b}$ is writing as $\mathcal{D}_2[V^a, \mathrm{b}]$

```
𝒟₁[Vᵃ, b] // EvaExp
𝒟₂[Vᵃ, b] // EvaExp
```

$V^{a}_{,b}$

$V^{a}_{,b} + V^{c}\,\Gamma^{a}_{cb}$

ReplaceIndices

Use ReplaceIndices Head to replace indices of equations

```
Vᵃ // EvaExp
Vᵃ // EvaExp // ReplaceIndices[#, {a → 1}] &
```

V^{a}

V^{1}

Calculate

Calculate Head uses the sixth element of Define evaluate tensor

```
Define[{g, {-μ, -ν}, , , , DiagonalMatrix[{1, X₁², X₁² Sin[X₂]²}]}]
g_μν // EvaExp
g_μν // EvaExp // ReplaceIndices[#, {μ → 2, ν → 2}] &
g_μν // EvaExp // ReplaceIndices[#, {μ → 2, ν → 2}] & // Calculate
```

$g_{\mu\nu}$

g_{22}

X_1^2

Example. Riemann Tensor

Now we can define important Tensors like the Riemann Tensor

```
Define[{R, {ρ, -μ, -σ, -ν}, 𝒟1[Γ^ρ_μν, σ] - 𝒟1[Γ^ρ_μσ, ν] + Γ^κ_μν Γ^ρ_κσ - Γ^κ_μσ Γ^ρ_κν}]
R^a_abc // EvaExp
```

$-\Gamma^{a}_{ab,c} + \Gamma^{a}_{ac,b} + \Gamma^{d}_{ac}\,\Gamma^{a}_{db} - \Gamma^{d}_{ab}\,\Gamma^{a}_{dc}$

Example. Ricci Tensor

```
Define[{R, {ρ, -μ, -σ, -ν}, 𝒟1[Γ^ρ_μν, σ] - 𝒟1[Γ^ρ_μσ, ν] + Γ^κ_μν Γ^ρ_κσ - Γ^κ_μσ Γ^ρ_κν}]
Define[{R, {-a, -b}, R^c_acb}]
R_ab // EvaExp
```

$\Gamma^{c}_{ab,c} - \Gamma^{c}_{ac,b} - \Gamma^{d}_{ac}\,\Gamma^{c}_{db} + \Gamma^{d}_{ab}\,\Gamma^{c}_{dc}$

Example. Commutator $[D_a, D_b]\, V^c = R^{a}_{dbc}\, V^d + 2\, S^{d}_{ab}\, D_d\, V^c$

```
Define[{R, {ρ, -μ, -σ, -ν}, 𝒟1[Γ^ρ_μν, σ] - 𝒟1[Γ^ρ_μσ, ν] + Γ^κ_μν Γ^ρ_κσ - Γ^κ_μσ Γ^ρ_κν}]
Define[{S, {ρ, -μ, -ν}, (Γ^ρ_μν - Γ^ρ_νμ)/2}]
Define[{Γ, {ρ, -μ, -ν}}]
V^d R^c_dab + 2 S^d_ab 𝒟2[V^c, d] // EvaExp
𝒟2[𝒟2[V^c, b], a] - 𝒟2[𝒟2[V^c, a], b] // EvaExp
```

$V^{c}_{,d}\,\Gamma^{d}_{ab} - V^{c}_{,d}\,\Gamma^{d}_{ba} - V^d\,\Gamma^{c}_{da,b} + V^d\,\Gamma^{c}_{db,a} + V^d\,\Gamma^{e}_{db}\,\Gamma^{c}_{ea} - V^d\,\Gamma^{e}_{da}\,\Gamma^{c}_{eb} + V^e\,\Gamma^{d}_{ab}\,\Gamma^{c}_{ed} - V^e\,\Gamma^{d}_{ba}\,\Gamma^{c}_{ed}$

$V^{c}_{,d}\,\Gamma^{d}_{ab} - V^{c}_{,d}\,\Gamma^{d}_{ba} - V^d\,\Gamma^{c}_{da,b} + V^d\,\Gamma^{c}_{db,a} + V^d\,\Gamma^{e}_{ab}\,\Gamma^{c}_{de} - V^d\,\Gamma^{e}_{ba}\,\Gamma^{c}_{de} + V^d\,\Gamma^{e}_{db}\,\Gamma^{c}_{ea} - V^d\,\Gamma^{e}_{da}\,\Gamma^{c}_{eb}$

Example. Commutator $[D_a, D_b]\, V^c = R^{a}_{dbc}\, V^d$ (If $\Gamma^{\rho}_{\mu\nu}$ is symmetric with respect to μ and ν)

```
Define[{R, {ρ, -μ, -σ, -ν}, 𝒟1[Γ^ρ_μν, σ] - 𝒟1[Γ^ρ_μσ, ν] + Γ^κ_μν Γ^ρ_κσ - Γ^κ_μσ Γ^ρ_κν}]
Define[{S, {ρ, -μ, -ν}, (Γ^ρ_μν - Γ^ρ_νμ)/2}]
Define[{Γ, {ρ, -μ, -ν}, , {{-μ, -ν}}}]
V^d R^c_dab // EvaExp
𝒟2[𝒟2[V^c, b], a] - 𝒟2[𝒟2[V^c, a], b] // EvaExp // SimplifyTensor
```

$-V^d\,\Gamma^{c}_{da,b} + V^d\,\Gamma^{c}_{db,a} + V^d\,\Gamma^{e}_{db}\,\Gamma^{c}_{ea} - V^d\,\Gamma^{e}_{da}\,\Gamma^{c}_{eb}$

$-V^d\,\Gamma^{c}_{da,b} + V^d\,\Gamma^{c}_{db,a} + V^d\,\Gamma^{e}_{db}\,\Gamma^{c}_{ea} - V^d\,\Gamma^{e}_{da}\,\Gamma^{c}_{eb}$

Example. Spherical Metric

```
Clear["Global`*"];
Get["C:\\Library\\Mathematica\\Afak Publish\\ATP.m"];
SetSystem[3 , X ]
Define[{R, {-a, -b}, R^c_acb}]
Define[{R, {ρ, -μ, -σ, -ν}, 𝒟1[Γ^ρ_μν, σ] - 𝒟1[Γ^ρ_μσ, ν] + Γ^κ_μν Γ^ρ_κσ - Γ^κ_μσ Γ^ρ_κν}]
Define[{g, {-μ, -ν}, , , , DiagonalMatrix[{1, X1^2, X1^2 Sin[X2]^2}]}]
Define[{g, {μ, ν}, , , , Inverse[DiagonalMatrix[{1, X1^2, X1^2 Sin[X2]^2}]]}]
Define[{Δ, {μ, -ν, -κ}, 1/2 g^μλ (𝒟1[g_λν, κ] + 𝒟1[g_λκ, ν] - 𝒟1[g_νκ, λ])}]
Define[{Γ, {μ, -ν, -κ}, , , , Δ^a_bc // EvaExp // TableTensor[#, {a, b, c}] &}]

R_bc // EvaExp // TableTensor[#, {b, c}] & // FullSimplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$